SyDe312 (Winter 2005)

Unit 1 - Solutions

February 3, 2005

Chapter 6 - Linear Systems

Problem 6.1 - 4

$$A = \begin{bmatrix} 3 & 1 & 0 & 0 & ... & ... & 0 \\ 1 & 3 & 1 & 0 & ... & ... & 0 \\ 0 & 1 & 3 & 1 & 0 & ... & 0 \\ \vdots & & \ddots & & & & \\ 0 & ... & ... & 0 & 1 & 3 & 1 \\ 0 & ... & ... & 0 & 0 & 1 & 3 \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_j \\ \vdots \\ x_n \end{bmatrix} \quad b = \begin{bmatrix} 4 \\ 5 \\ 5 \\ \vdots \\ 5 \\ 4 \end{bmatrix}$$

$$\begin{aligned} 3x_1 + 1x_2 + 0x_3 + cdots + 0x_j + \cdots + 0x_n &= 4 \\ 1x_1 + 3x_2 + 1x_3 + 0x_4 + \cdots + \cdots + 0x_n &= 5 \\ 0x_1 + 1x_2 + 3x_3 + 1x_4 + 0x_5 + \cdots + 0x_n &= 5 \\ &\cdots \\ 0x_1 + \cdots + 1x_{j-1} + 3x_j + 1x_{j+1} + \cdots + 0x_n &= 5 \\ &\cdots \\ 0x_1 + \cdots + 0x_j + \cdots + 0x_{n-2} + 1x_{n-1} + 3x_n &= 4 \end{aligned}$$

So, the linear system of order n is:

$$\begin{aligned} 3x_1 + x_2 &= 4 \\ x_{j-1} + 3x_j + x_{j+1} &= 5 \quad j = 2, 3, \ldots, n-1 \\ x_{n-1} + 3x_n &= 4 \end{aligned}$$

Problem 6.1 - 5

```
A=zeros(n);
for i=1:n
   for j=1:n
      if i<j
         A(i,j)=i/j;
      else
         A(i,j)=j/i;
      end
   end
end
b=zeros(n,1);
for i=1:n
   b(i,1)=(-1)^(-(i+1));
end
```

Problem 6.1 - 6

```
beta=1;
A=zeros(n);
for j=1:n
   A(1,j)=beta
   A(j,1)=beta
end
for i=2:n
   for j=2:n
      A(i,j)=A(i-1,j)+A(i,j-1)
   end
end
b=zeros(n,1);
for i=1:n
   b(i,1)=(-1)^(i-1)/i;
end
```

Problem 6.2 - 7

From problem 5: if $A = ww^T = 1 \quad \Rightarrow A^2 = A = 1$

From problem 6: if $B = B^T$, so: $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad B^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \Rightarrow b = c$

$\Rightarrow B = \begin{bmatrix} a & b \\ b & d \end{bmatrix}$

Now, define B as $\quad B = I - 2ww^T$,

$B^T = I^T - 2(ww^T)^T = I - 2(w^T)^T w^T = I - 2ww^T = B$,

(since $(A^T)^T = A$ and $(AB)^T = B^T A^T$)

$B^2 = (I - 2ww^T)^2 = I - 4ww^T + 4(ww^T)^2$,

$= I - 4ww^T + 4ww^T$, using problem 5.

$= I$

since $B^2 = B.B = I$, and $B.B^{-1} = I$, so: $B^{-1} = B$

Problem 6.2 - 9

Associative law for matrix multiplication: $(AB)C = A(BC)$,

so, $x = (u^T v)w^T = u^T(vw^T)$,

where, $[u]_{n\times1}$, $[u^T]_{1\times n}$, $[v]_{n\times1}$, $[w]_{n\times1}$, $[w^T]_{1\times n}$,and

$[u^T]_{1\times n} \times [v]_{n\times1} \times [w^T]_{1\times n} = [x]_{n\times1}$,

for $x = (u^T v)w^T$,

$$[u_1 ... u_n] . \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \quad \rightarrow n \text{ multiplications and } (n-1) \text{ additions.}$$

$$\text{again } \times \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix} \quad \rightarrow \text{another } n \text{ multiplications.}$$

So, $n + n = 2n$ multiplications and $n - 1$ additions .

for $x = u^T(vw^T)$,

$$\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} .[w_1 ... w_n] \rightarrow \text{a matrix of size } (n \times n) \rightarrow n^2 \text{ multiplications.}$$

$[u_1 u_2 ... u_n].(vw^T) \rightarrow$ a matrix of size $(1 \times n) \rightarrow$ another n^2 multiplications and $n(n-1)$ additions..

So, $n^2 + n^2 = 2n^2$ multiplications and $n(n-1)$ additions .

Obviously the first way is preferable.

Problem 6.2 - 10

$[A]_{m \times n}$, $[B]_{n \times p}$, $[C]_{p \times q}$, $[AB]_{m \times p}$, $[BC]_{n \times q}$, $[ABC]_{m \times q}$

$[(AB)C]_{ij} = \sum_{l=1}^{p} (AB)_{il} C_{lj} = \sum_{l=1}^{p} [\sum_{k=1}^{n} A_{ik} B_{kl}] C_{lj}$

(Multiplication/Division)$MD[AB] = MD[\quad []_{m \times n} []_{n \times p} \quad] = mnp$,

likewise, $MD[\quad []_{m \times p} []_{p \times q} \quad] = mpq$, So, $MD[(AB)C] = mnp + mpq = mp(n+q)$.

$[A(BC)]_{ij} = \sum_{k=1}^{n} A_{ik}(BC)_{kj} = \sum_{k=1}^{n} A_{ik}[\sum_{l=1}^{p} B_{kl}C_{lj}]$

$MD[BC] = MD[\quad []_{n\times p}[]_{p\times q} \quad] = npq,$

and, $MD[\quad []_{m\times n}[]_{n\times q} \quad] = mnq,$

So, $MD[A(BC)] = npq + mnq = nq(p+m)$.

If p=1, and q=m=n=100. then,

$MD[(AB)C] = 100(100+100) = 20000, \quad MD[A(BC)] = 100 \times 100(101) = 1010000.$

Thus the product $(AB)C$ is generally much less expensive than the product $A(BC)$.

Problem 6.3 - 1

The modified Matlab program is :

```
function [x,lu,piv]= newGEpivot (A,b)
[m,n]=size(A);
if m~=n
   error('The matrix is not square. ');
end
m=length(b);
if m~=n
    error('The matrix and the vecttor do not match in size. ');
end
piv = (1:n)';
for k=1:n-1
   A(k+1:n,k)=A(k+1:n,k)/A(k,k);
   for i= k+1:n
      A(i,k+1:n)= A(i,k+1:n)-A(i,k)*A(k,k+1:n);
   end
   b(k+1:n)= b(k+1:n) - A(k+1:n,k)*b(k);
```

```
end

x=zeros(n,1);
x(n)=b(n)/A(n,n);
for i = n-1 : -1 : 1
   x(i)= (b(i)-A(i,i+1:n)*x(i+1:n))/A(i,i);
end
lu=A;
```

And using this program the results are:
(a) $x_1 = 1, \quad x_2 = 2, \quad x_3 = -2.$
(b) $x_1 = -2.3750, \quad x_2 = 4.2500, \quad x_3 = -0.5000, \quad x_4 = -1.0000.$
(c) $x_1 = -3, \quad x_2 = -2, \quad x_3 = 1.$

Problem 6.3 - 2

This is a straightforward use of GEpivot. The matrix of coefficients is a well-conditioned matrix, and the solution of the linear system should be very accurate.

For example for n = 5, we have:

```
n=5;
A= ones(n);
for i =1 : n
   for j = 1 : n
      A(i,j)= max(i,j);
   end
end
b= ones(n,1);
[x,lu,piv]=GEpivot(A,b);
x
```

The result will be:

$$x = [-0.0000 \quad 0.0000 \quad -0.0000 \quad 0.0000 \quad 0.2000]^T$$

Problem 6.3 - 5

Problem 6.3 - 5a

```
A = [5 7 6 5;7 10 8 7; 6 8 10 9; 5 7 9 10];
b = [1 -1 -1 1]';
```

Using GEpivot,

```
format long
[x,lu,piv]=GEpivot(A,b);
x
```

the result will be:

$$x = 1.0e002 * \begin{bmatrix} 1.35999999999997 \\ -0.81999999999998 \\ -0.34999999999999 \\ 0.20999999999999 \end{bmatrix},$$

but using GEdemo,

```
x=GEdemo(A,b,16)
```

the result is:

$$x = 1.0e002 * \begin{bmatrix} 1.36000000000000 \\ -0.82000000000000 \\ -0.35000000000000 \\ 0.21000000000000 \end{bmatrix},$$

Interestingly, using GEdemo with severely reduced precision:

```
x=GEdemo(A,b,3)
```

gives an apparently more accurate result:

$$x = [136, \quad -82, \quad -35, \quad 21]$$

however this is illusory, since only the integer portions of these numbers are significant. By coincidence, the exact solution happens to be integers.

Problem 6.3 - 5b

Using GEpivot:

$$x = 1.0e004 * \begin{bmatrix} 0.05160000000000 \\ -0.56999999999995 \\ 1.36199999999987 \\ -0.88199999999992 \end{bmatrix},$$

and with GEdemo:

$$x = 1.0e004 * [0.0516, \quad -0.5700, \quad 1.3620, \quad -0.8820]$$

Problem 6.5 - 1

The program is shown below:

```
function [x,r,e,s,t]= SOLEQ(A,b)
x=GEpivot(A,b);
r=b-A*x;
e=GEpivot(A,r);
s=x+e;
norm(e)/norm(x);
```

Problem 6.5 - 1a

$$\begin{aligned}
\hat{x} &= [0.02000000, -0.03000000, 0.05000000, 0.03000000]^T \\
r &= [0.05551115E-15, 0.0, 0.11102230E-15, 0.02775557E-15]^T \\
\hat{e} &= [0.13877787E-15, 0.27755575E-15, 0.44408920E-15, 0.61062266E-15]^T \\
\hat{x}+\hat{e} &= [0.02000000, -0.03000000, 0.05000000, 0.03000000]^T \\
\frac{\|\hat{e}\|}{\|\hat{x}\|} &= 1.19E-14
\end{aligned}$$

Problem 6.5 - 1b

$$\begin{aligned}
\hat{x} &= [0.99999999, 1.00000000, 1.00000000, 1.00000000]^T \\
r &= [0.35527136E-14, 0.0, 0.0, 0.0]^T
\end{aligned}$$

$$\hat{e} = [0.24158453E-12, -0.1456612E-12, -0.06039613E-12, 0.03552713E-12]^T$$
$$\hat{x}+\hat{e} = [1.00000000, 0.99999999, 0.99999999, 1.00000000]^T$$
$$\frac{\|\hat{e}\|}{\|\hat{x}\|} = 1.45E-13$$

Problem 6.5 - 1c

$$\hat{x} = [33.999999, -695.999999, 3191.999999, -5012.000000, 2520.000000]^T$$
$$r = [-0.056843E-12, -0.193289E-12, -0.090927E-12, -0.068223E-12]^T$$
$$\hat{e} = [-0.006595E-8, 0.136151E-8, -0.615430E-8, 0.955502E-8, -0.476088E-8]^T$$
$$\hat{x}+\hat{e} = [33.999999, -695.999999, 3191.999999, -5011.000000, 2519.000000]^T$$
$$\frac{\|\hat{e}\|}{\|\hat{x}\|} = 1.91E-12$$

This is a very ill-conditioned linear system. Iterative improvement (residual correction) will fix the solution (Try it!).

Problem 6.5 - 3

$$A = \begin{bmatrix} 5 & 7 \\ 7 & 10 \end{bmatrix}, \qquad A^{-1} = \begin{bmatrix} 10 & -7 \\ -7 & 10 \end{bmatrix}$$
$$\|A\|_1 = 17, \quad \left\|A^{-1}\right\|_1 = 17, \quad \mathrm{cond}(A) = 289$$

The row sum norm has been used in the calculation above for simplicity.

The two equations in (6.86) can be rewritten as,

$$x_1 + 1.4x_2 = 0.14$$
$$x_1 + 10/7x_2 = 1/7$$

These have graphs that are almost parallel straight lines. Changing the right side by a small amount means changing the y-intercepts of these lines by a small amount. But because they are almost parallel, it changes their intersections by a relatively large amount.

Problem 6.5 - 4

$$A = \begin{bmatrix} 19 & 20 \\ 20 & 21 \end{bmatrix}, \qquad \|A\|_1 = 41$$

$$A^{-1} = \begin{bmatrix} -21 & 20 \\ 20 & -19 \end{bmatrix}, \qquad \left\|A^{-1}\right\|_1 = 41$$

$$\text{cond}(A) = \|A\| \left\|A^{-1}\right\| = 41 \times 41 = 1681$$

As the condition number is very large, the system is ill-conditioned with respect to perturbations of the right-hand side constants $\{b_1, b_2\}$.

Problem 6.4 - 1

We illustrate hand calculation of an LU decomposition with strict partial pivoting, using two different approaches to the permutation book-keeping. A strict partial pivoting scheme requires that the largest entry in absolute value is always to be used as the pivot.

The system to be solved is:

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 2 & 3 \\ -1 & -3 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix}$$

Method 1: Direct Gaussian elimination, P on left hand side

We construct the decomposition directly as $A = PLU$. Both P and L start as identity matrices and U starts as A.

$$PLU = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 2 & 2 & 3 \\ -1 & -3 & 0 \end{bmatrix}$$

$R_2 \leftrightarrow R_1$
This give $PLP'U$. Moving the P' matrix through L has no effect, since L is still the identity matrix. This gives $PP'LU$. The P' on the right of P has the effect of interchanging columns 1 and 2 of P. The result is:

$$PLU = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 2 & 3 \\ 1 & 2 & 1 \\ -1 & -3 & 0 \end{bmatrix}$$

$R_2 \leftarrow R_2 - (1/2)R_1$
$R_3 \leftarrow R_3 + (1/2)R_1$

$$PLU = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ -1/2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 2 & 3 \\ 0 & 1 & -1/2 \\ 0 & -2 & 3/2 \end{bmatrix}$$

$R_2 \leftrightarrow R_3$
This gives $PLP'U$. Moving the P' matrix through L interchanges columns 2 and 3, then rows 2 and 3 of L. This gives $PP'LU$. The matrix P' can be combined with P by interchanging columns 2 and 3 of P. The result is:

$$PLU = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -1/2 & 1 & 0 \\ 1/2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 2 & 3 \\ 0 & -2 & 3/2 \\ 0 & 1 & -1/2 \end{bmatrix}$$

$R_3 \leftarrow R_3 + (1/2)R - 2$

$$PLU = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -1/2 & 1 & 0 \\ 1/2 & -1/2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 2 & 3 \\ 0 & -2 & 3/2 \\ 0 & 0 & 1/4 \end{bmatrix}$$

Method 2: Direct Gaussian elimination, P on right hand side

We construct the decomposition directly as $PA = LU$. The P is kept track of separately. The work can be done in situ.

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 2 & 3 \\ -1 & -3 & 0 \end{bmatrix} \cdots P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_2 \leftrightarrow R_1$

$$\begin{bmatrix} 2 & 2 & 3 \\ 1 & 2 & 1 \\ -1 & -3 & 0 \end{bmatrix} \cdots P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_2 \leftarrow R_2 - (1/2)R_1$
$R_3 \leftarrow R_3 + (1/2)R_1$

$$\left[\begin{array}{c|cc} \multicolumn{1}{c}{2} & 2 & 3 \\ \hline 1/2 & 1 & -1/2 \\ -1/2 & -2 & 3/2 \end{array}\right] \cdots P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_3 \leftrightarrow R_3$

$$\left[\begin{array}{c|cc} \multicolumn{1}{c}{2} & 2 & 3 \\ \hline -1/2 & -2 & 3/2 \\ 1/2 & 1 & -1/2 \end{array}\right] \cdots P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$R_3 \leftarrow R_3 + (1/2)R_2$

$$\begin{bmatrix} 2 & 2 & 3 \\ -1/2 & -2 & 3/2 \\ 1/2 & -1/2 & 1/4 \end{bmatrix} \cdots P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

This would be used in the form $LUx = Pb$. The result is the same as last time if we multiply on the left by $P^{-1} = P^T$):

$$P^T A = P^T LU \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -1/2 & 1 & 0 \\ 1/2 & -1/2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 2 & 3 \\ 0 & -2 & 3/2 \\ 0 & 0 & 1/4 \end{bmatrix}$$

Problem 6.4 - 2

Problem 6.4 - 2a

LU factorization of $A = \begin{bmatrix} 2 & 1 & -1 \\ 4 & 0 & -1 \\ -8 & 2 & 2 \end{bmatrix}, \quad b = \begin{bmatrix} 6 \\ 6 \\ -8 \end{bmatrix}$

$LU = A$

$$\begin{bmatrix} 1 & 0 & 0 \\ m_{21} & 1 & 0 \\ m_{31} & m_{32} & 1 \end{bmatrix} \times \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} = \begin{bmatrix} 2 & 1 & -1 \\ 4 & 0 & -1 \\ -8 & 2 & 2 \end{bmatrix}$$

Use Doolittle's method (Crout's algorithm):

First row multiplication:
$u_{11} = a_{11} = 2, \quad u_{12} = a_{12} = 1, \quad u_{13} = a_{13} = -1$

Second row multiplication:
$m_{21} = \frac{a_{21}}{u_{11}} = 2, \quad u_{22} = a_{22} - m_{21}u_{12} = -2, \quad u_{23} = a_{23} - m_{21}u_{13} = 1$

Third row multiplication:
$m_{31} = \frac{a_{31}}{u_{11}} = -4, \quad m_{32} = (a_{32} - m_{31}u_{12})/u_{22} = -3,$
$u_{33} = a_{33} - m_{31}u_{13} - m_{32}u_{23} = 1$

Therefore: $L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -4 & -3 & 1 \end{bmatrix}, U = \begin{bmatrix} 2 & 1 & -1 \\ 0 & -2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

Problem 6.4 - 2b

Use the Matlab function

```
[L,U,P]=lu(A)
```

$$A = \begin{bmatrix} 2 & 1 & -1 & -2 \\ 4 & 4 & 1 & 3 \\ -6 & -1 & 10 & 10 \\ -2 & 1 & 8 & 4 \end{bmatrix}, b = \begin{bmatrix} 2 \\ 4 \\ -5 \\ 1 \end{bmatrix}$$

$$L = \begin{bmatrix} 1.0000 & 0 & 0 & 0 \\ -0.6667 & 1.0000 & 0 & 0 \\ 0.3333 & 0.4000 & 1.0000 & 0 \\ -0.3333 & 0.2000 & 0.5000 & 1.0000 \end{bmatrix}$$

$$U = \begin{bmatrix} -6.0000 & -1.0000 & 10.0000 & 10.0000 \\ 0 & 3.3333 & 7.6667 & 9.6667 \\ 0 & 0 & 1.6000 & -3.2000 \\ 0 & 0 & 0 & 1.0000 \end{bmatrix}$$

We verify that $LU = PA$

$$LU = PA = \begin{bmatrix} -6 & -1 & 10 & 10 \\ 4 & 4 & 1 & 3 \\ -2 & 1 & 8 & 4 \\ 2 & 1 & -1 & -2 \end{bmatrix},$$

where P records the row interchanges used during the factorization: $P = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$

Problem 6.4 - 2c

$$A = \begin{bmatrix} 1 & -1 & 2 \\ -1 & 5 & 4 \\ 2 & 4 & 29 \end{bmatrix}, b = \begin{bmatrix} 1 \\ -3 \\ 15 \end{bmatrix}$$

$$L = \begin{bmatrix} 1.0000 & 0 & 0 \\ -0.5000 & 1.0000 & 0 \\ 0.5000 & -0.4286 & 1.0000 \end{bmatrix},$$

$$U = \begin{bmatrix} 2.0000 & 4.0000 & 29.0000 \\ 0 & 7.0000 & 18.5000 \\ 0 & 0 & -4.5714 \end{bmatrix}$$

Verify that $LU = PA$

$$LU = PA = \begin{bmatrix} 2 & 4 & 29 \\ -1 & 5 & 4 \\ 1 & -1 & 2 \end{bmatrix},$$

where P records the row interchanges used during the factorization: $P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

Problem 6.4 - 5

Problem 6.4 - 5a

We know that $A = LU$.
D is a non-singular matrix so we can say there exist D^{-1}, such that $DD^{-1} = I$.

$$A = LU = LIU = LDD^{-1}U = \underbrace{LD}_{L_1}\underbrace{D^{-1}U}_{U_1} = L_1U_1$$

So, LU factorization is not unique.

Problem 6.4 - 5b

Note that all these matrices are non-singular. Thus,

$$\det A = \det L_1 \det U_1 = \det L_2 \det U_2 \neq 0$$

$$\Rightarrow \text{The inverse matrices exist.}$$

$$\text{Multiply both sides of } L_1U_1 = L_2U_2 \text{ by } L_2^{-1}(\quad)U_1^{-1}$$

$$L_2^{-1}L_1U_1U_1^{-1} = L_2^{-1}L_2U_2U_1^{-1}$$

$$L_2^{-1}L_1 = U_2U_1^{-1}$$

Since L_2 is lower triangular, L_2^{-1} is also triangular. The result of $L_2^{-1}L_1$ and $U_1U_1^{-1}$ will be triangular, too. The same is for the right side, unless they are upper triangular matrices. Since these two products are equal, they must be diagonal matrices. This fact requires D to be a diagonal matrix.

Problem 6.4 - 6

Problem 6.4 - 6a

Cholesky factorization of symmetric matrices:

$$A = LL^T$$

(a)

$$A = \begin{bmatrix} 1 & -1 \\ -1 & 5 \end{bmatrix}, L = \begin{bmatrix} l_{11} & 0 \\ l_{21} & l_{22} \end{bmatrix}$$

$$LL^T = \begin{bmatrix} a & 0 \\ b & c \end{bmatrix} \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 5 \end{bmatrix}$$

$$\begin{bmatrix} a^2 & ab \\ ab & b^2 + c^2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 5 \end{bmatrix}$$

$$a \text{ and } c \text{ should be positive, } \Rightarrow a = 1, b = -1, c = 2.$$

Problem 6.4 - 6b

$$A = \begin{bmatrix} 2.25 & -3 & 4.5 \\ -3 & 5 & -10 \\ 4.5 & -10 & 34 \end{bmatrix}$$

Similarly we get L as, $L = \begin{bmatrix} 1.5 & 0 & 0 \\ -2.0 & 1 & 0 \\ 3.0 & -4 & 3 \end{bmatrix}$

Problem 6.4 - 8

Using the provided m-file `tridiag`

```
A=[ 2 1 0 0 0;1 2 1 0 0;0 1 2 1 0;0 0 1 2 1;0 0 0 1 2];
f=[1 0 0 0 0 0]';
a=[0,1,1,1,1];
b=[2,2,2,2,2];
c=[1,1,1,1];
n=5;
[x, ier, alpha, beta]=tridiag (a,b,c,f,n,iflag);
x
```

The result will be: $x = [0.8333, -0.6667, 0.5000, -0.3333, 0.1667]^T$.

Problem 6.4 - 10

As in previous problem, using `tridiag`

```
n=100;
f=ones(n,1);
a=ones(n,1);
c=a;
a(1)=0;
c(n)=0;
b=4*f;
iflag=0;
[x, ier, alpha, beta]=tridiag (a,b,c,f,n,iflag);
```

```
x
```

The result will be:

$x = [0.2113 \quad 0.1547 \quad 0.1699 \quad 0.1658 \quad 0.1669 \quad 0.1666 \quad 0.1667 \quad 0.1667 \quad ...$
$0.1667 \quad 0.1667 \quad 0.1667 \quad 0.1666 \quad 0.1669 \quad 0.1658 \quad 0.1699 \quad 0.1547 \quad 0.2113]^T.$